The Bochner Method

Broadly stated, the Bochner Technique refers to a family of methods for obtaining what are often topological conclusions about a manifold using information about curvature and differential operators that live on the manifold.

In particular, we have several different ideas about what a Laplacian is on a Riemannian manifold, and since the Laplacian is arguably the single most important differential operator, it seems reasonable that we should try to study how different incarnations of it might interact.

References:

- The Bochner Technique in Differential Geometry – Hung-Hsi Wu
- Riemannian Geometry – Peter Petersen
- Riemannian Geom. and Geom. Analysis – Jurgen Jöst
- Einstein Manifolds – Arthur Besse

The Weitzenböck Formulas:

Since we apparantly have two competiting notions of a Laplacian on M, we should try to figure out how they relate.

To this end, it will be useful to cook up some technical results that relate the differential operators d and δ to ∇ and R^∇.

Lemma 1: Let $\{V_i\}$ be an ON frame for TM near $x \in M$, and $\{\omega^i\}$ its dual co-frame. Then

(i) $d = \omega^i \wedge \nabla_{V_i}$

(ii) $\delta = -\sum_j i(V_j)\nabla_{V_j}$

Note: $i(-)$ is the interior multiplication operator, which acts on p-forms by sending $\omega \longmapsto i(v)\omega$ (a p-1 form) defined by

$$[i(v)\omega](X_1,\ldots,X_{p-1}) := \omega(V, X_1, \ldots, X_{p-1})$$

Proof: This technique is generally quite useful, so let's go over it carefully:

Step 1: Show that the expression at hand is invariant with respect to the choice of coordinates, local frame, etc.

Step 2: Prove the formula using a conventient choice of coordinates or local frame. Often the following is useful:

Lemma: Given $x \in M$ and an ON base $\{v_i\}$ of T_xM, $\exists$ ON local frame $\{V_i\}$ for TM which is normal at x with $V_i(p) = v_i$.

Here, normal at x means that $\nabla_{V_i}V_j(x) = 0$. Being torsion free implies further that $[V_i, V_j](x) = 0$.

Remark: Geodesic normal coordinates centered at x satisfy all of the above, except that they need not induce everywhere ON vector fields.

Lets apply this technique to exhibiting the proposed formulas for d and δ:

First, let $d_0 := \omega^i \wedge \nabla_{v_i}$ and $\delta_0 = -\sum_j i(v_j)\nabla_{v_j}$. We aim to show that $d = d_0$ and $\delta = \delta_0$, and we begin by observing that if $\{w_i\}$ is any other local ON frame about x with $\{\eta^i\}$ its dual coframe, then

$$\omega^i \wedge \nabla_{v_i} = d_0 = \eta^i \wedge \nabla_{w_i}$$

and

$$-\sum_j i(v_j)\nabla_{v_j} = \delta_0 = -\sum_j i(w_j)\nabla_{w_j}$$

This is easy to check by writing $w_i = a_i^{\ j} v_j$ and $\eta^i = b^i_{\ j}\omega^j$ for smooth functions $a_i^{\ j}, b^i_{\ j}$ defined on the common domain of $\{v_i\}, \{w_i\}$. Note the important facts that $a_i^{\ j} b_j^{\ k} = \delta_i^{\ k}$ and that (a_i^j) is orthogonal.

We now show the validity of the claimed formulas at $x \in M$. Since p was arbitrary, the claim will then follow. Let (x^i) be normal coordinates at x, and set $v_i = \partial_{x^i}$, $\omega^i = dx^i$. Since d, d_0, δ, δ_0 are linear, it suffices to prove the formulas where the argument has the form $f\omega^1\wedge\cdots\wedge\omega^p$ (re ordering the basis if necessary). For d_0, we compute that

$$\begin{aligned}
d_0(f\omega^1\wedge\cdots\wedge\omega^p) &= (\omega^i \wedge \nabla_{v_i})(f\omega^1\wedge\cdots\wedge\omega^p)\\
&= \omega^i \wedge \left\{ \nabla_{v_i}(f)\,\omega^1\wedge\cdots\wedge\omega^p + f\,\nabla_{v_i}(\omega^1\wedge\cdots\wedge\omega^p)\right\}\\
&= \omega^i \wedge \left\{ v_i(f)\,\omega^1\wedge\cdots\wedge\omega^p + f\sum_j \omega^1\wedge\cdots\wedge\nabla_{v_i}\omega^j\wedge\cdots\wedge\omega^p\right\}
\end{aligned}$$

and so at x where $(\nabla_{v_i} v_j)(x) = (\nabla_{v_i}\omega^j)(x) = 0$, we get

$$d_0(f\omega^1\wedge\cdots\wedge\omega^p)|_x = (v_i f)\,\omega^i\wedge\omega^1\wedge\cdots\wedge\omega^p|_x = d(f\omega^1\wedge\cdots\wedge\omega^p)|_x .$$

For δ_0, we compute that

$$\begin{aligned}
\delta_0(f\omega^1\wedge\cdots\wedge\omega^p)|_x &= -\sum_j i(v_j)\nabla_{v_j}(f\omega^1\wedge\cdots\wedge\omega^p)|_x\\
&= -\sum_j i(v_j)\left\{ v_j(f)\omega^1\wedge\cdots\wedge\omega^p + f\sum_i \omega^1\wedge\cdots\wedge\widehat{\nabla_{v_j}\omega^i}\wedge\cdots\omega^p\right\}\Big|_x\\
&= -\sum_j i(v_j)\, v_j(f)\omega^1\wedge\cdots\wedge\omega^p|_x\\
&= -\sum_j v_j(f)(-1)^{k-1}\delta_j^{\ k}\,\omega^1\wedge\cdots\wedge\widehat{\omega^k}\wedge\cdots\wedge\omega^p|_x\\
&= -\sum_j (-1)^{j-1} v_j(f)\,\omega^1\wedge\cdots\wedge\widehat{\omega^j}\wedge\cdots\wedge\omega^p|_x
\end{aligned}$$

Meanwhile, $\delta = (-1)^{n(p+1)+1} * d *$, so also at x we have

$$\begin{aligned}
\delta(f\omega^1\wedge\cdots\wedge\omega^p) &= (-1)^{n(p+1)+1} * d(f\omega^{p+1}\wedge\cdots\wedge\omega^n) = (-1)^{n(p+1)+1} * (\omega^i\wedge\nabla_{v_i}(f\omega^{p+1}\wedge\cdots\wedge\omega^n))\\
&= (-1)^{n(p+1)+1} v_i(f) * \omega^i\wedge\omega^{p+1}\wedge\cdots\wedge\omega^n\\
&= \sum_i (-1)^{n(p+1)+1}\,\mathrm{sgn}((i)(p+1)\cdots(n)\ (1)\cdots\widehat{(i)}\cdots(p))\, v_i(f)\,\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\omega^p\\
&= \sum_i (-1)^{n(p+1)+1+(n-p+1)(p-1)+p-i}\, v_i(f)\,\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^p\\
&= \sum_i (-1)^i\, v_i(f)\,\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^p.
\end{aligned}$$

Weitzenböck Formula I:

Let M^n be an oriented Riem. mfd., $\{V_i\}$ a local ON frame, and $\{\omega^i\}$ its dual co-frame. Then

$$\Delta = -\Delta^\nabla - \Sigma_j\, \omega^i \wedge \iota(V_j)\, R(V_i, V_j)$$

where $\Delta = (d+\delta)^2 = d\delta + \delta d$ is the Hodge Laplacian on $\Omega^\bullet(M)$ and $-\Delta^\nabla = -\operatorname{tr}(\nabla^2) = \nabla^* \nabla$ is the connection Laplacian.

Thus, these two Laplace operators differ by some function of curvature.

Corollary to WF I: $\Delta = -\Delta^\nabla$ on $C^\infty(M)$ and on $\Omega^n(M)$.

Proof: That $R(V_i, V_j)$ annihilates $C^\infty(M)$ is clear since R is a tensor and so $R(V_i, V_j)f = fR(V_i, V_j)1 = 0$.

Let $\omega \in \Omega^n(M)$. Then $R(V_i, V_j)\omega =: \eta_{ij} = \varphi_{ij}\, \omega^1 \wedge \cdots \wedge \omega^n$ is also in $\Omega^n(M)$, and so

$$\begin{aligned}
\Sigma_j\, \omega^i \wedge \iota(V_j)\eta_{ij} &= \Sigma_j\, \omega^i \wedge \iota(V_j)\, \varphi_{ij}\omega^1 \wedge \cdots \wedge \omega^n \\
&= \Sigma_j\, \omega^i \wedge \left(\varphi_{ij} \Sigma_{k=1}^n (-1)^{k-1}\, \omega^k(V_j)\, \omega^1 \wedge \cdots \wedge \widehat{\omega^k} \wedge \cdots \wedge \omega^n\right) \\
&= \Sigma_j\, \omega^i \wedge \left((-1)^{j-1} \cdot \varphi_{ij}\, \omega^1 \wedge \cdots \wedge \widehat{\omega^j} \wedge \cdots \wedge \omega^n\right) \\
&= \varphi_{ii}\, \omega^1 \wedge \cdots \wedge \omega^n \\
&= R(V_i, V_i)\, \eta \equiv 0 .
\end{aligned}$$

Thus in either case, the curvature "error" term in WF I vanishes. ■

Proof of WF I: As a first step, check that both sides are independent of the choice of local ON frame $\{V_i\}$.

Next, we verify that the formula holds at an arbitrary $p \in M$, about which we fix a loc. ON normal frame.

At p, by normality,

- $\operatorname{tr} \nabla^2 = \Sigma\, \nabla_{V_i} \nabla_{V_i} - \nabla_{\nabla_{V_i} V_i} = \Sigma \nabla_{V_i} \nabla_{V_i}$
- $R(V_i, V_j) = [\nabla_i, \nabla_j] - \nabla_{[V_i, V_j]} = [\nabla_i, \nabla_j]$

Now we compute at p that (since $\nabla_{V_i}\omega^j(p)=0$)

$$\delta d = -\sum_j i(V_j)\nabla_{V_j}(\omega^i\wedge\nabla_{V_i})$$
$$= -\sum_j i(V_j)\{\nabla_{V_j}\omega^i\wedge\nabla_{V_i} + \omega^i\wedge\nabla_{V_j}\nabla_{V_i}\}$$
$$= -\sum_j i(V_j)\,\omega^i\wedge\nabla_{V_j}\nabla_{V_i}$$
$$= -\sum_i\{\nabla_{V_i}\nabla_{V_i} - \sum_j \omega^i\wedge i(V_j)\nabla_{V_j}\nabla_{V_i}\}$$
$$= -\operatorname{tr}\nabla^2 + \sum_j \omega^i\wedge i(V_j)\nabla_{V_j}\nabla_{V_i}$$

On the other hand, using that $i(V_j)\nabla_{V_k} = \nabla_{V_k} i(V_j)$ at p,

$$d\delta = -\omega^i\wedge\nabla_{V_i}\left(\sum_j i(V_j)\nabla_{V_j}\right)$$
$$= -\sum_j \omega^i\wedge i(V_j)\nabla_{V_i}\nabla_{V_j}.$$

Thus,
$$\Delta = d\delta + \delta d = -\operatorname{tr}\nabla^2 - \sum_j \omega^i\wedge i(V_j)R(V_i,V_j)$$

Using this formula, we can establish a few more very useful results. Recall that $\langle -,-\rangle$ on M induces a unique fibre metric on $\Omega^p(M)$ satisfying the formula

$$\langle \omega^1\wedge\cdots\wedge\omega^p, \eta^1\wedge\cdots\wedge\eta^p\rangle = \det(\langle\omega^i,\eta^j\rangle)$$

where $\langle\omega^i,\eta^j\rangle$ is the canonical fibre metric on $T^{(1,0)}TM$. Thus, in a local ON frame $\{V_i\}$ with dual co-frame $\{\omega^i\}$, we see that

$$\{\omega^{i_1}\wedge\cdots\wedge\omega^{i_p} : 1\le i_1<\cdots<i_p\le n\}$$

is a local ON base of $\Omega^p(M)$.

Weitzenböck Formula II: Let M^n be as above (cpt, oriented), and let $\phi\in\Omega^p(M)$. Then

$$-\tfrac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 - \langle\Delta\phi,\phi\rangle - F(\phi)$$

where

$$F(\phi) := \left\langle \sum_j \omega^i\wedge i(V_j)R(V_i,V_j)\phi\,,\,\phi\right\rangle$$

Corollary: If $\phi\in\Omega^p(M)$ is harmonic, then

$$-\tfrac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 - F(\phi)$$

Proof of WF II: We again begin by noting that the formula is invariant with respect to choice of ON local frame, so we pick an arbitrary point $p \in M$ and fix a local ON frame which is normal at p.

Beginning with **WF I**:

$$\Delta \phi = -\Delta^{\nabla}\phi - \Sigma_j \omega^i \wedge i(V_j) R(V_i, V_j)\phi$$

$$\Updownarrow$$

$$\langle \operatorname{tr}\nabla^2\phi, \phi\rangle = -\langle \Delta \phi, \phi\rangle - F(\phi)$$

Now compute that at p

$$\begin{aligned}
\langle \operatorname{tr}\nabla^2\phi, \phi\rangle &= \langle \Sigma_i \nabla_{V_i}\nabla_{V_i}\phi, \phi\rangle \\
&= \Sigma_i\{\nabla_{V_i}\langle \nabla_{V_i}\phi, \phi\rangle - |\nabla_{V_i}\phi|^2\} \\
&= \Sigma_i\{\tfrac{1}{2}\nabla_{V_i}\nabla_{V_i}|\phi|^2 - |\nabla_{V_i}\phi|^2\} \\
&= \tfrac{1}{2}\Delta^{\nabla}|\phi|^2 - |\nabla\phi|^2 \\
&= -\tfrac{1}{2}\Delta|\phi|^2 - |\nabla\phi|^2
\end{aligned}$$

Lemma: Let $\phi \in \Omega^1(M)$. Then $F(\phi) = -\operatorname{Ric}(\phi^{\#}, \phi^{\#})$

Proof: Fix a local ON frame $\{V_i\}$ with co-frame $\{\omega^i\}$. Then if $\phi = \phi_i\omega^i$, $\phi^{\#} = \Sigma_i \phi_i V_i$, so in particular

$$\phi^{\#} = \Sigma_i \langle \phi, \omega^i\rangle V_i$$

and thus

$$\begin{aligned}
F(\phi) &= \langle \Sigma_j \omega^i \wedge i(V_j) R(V_i, V_j)\phi, \phi\rangle \\
&= \Sigma_j\, i(V_j)R(V_i, V_j)\phi \langle \omega^i, \phi\rangle \\
&= -\Sigma_j\, \phi(R(V_i, V_j)V_j)\langle \omega^i, \phi\rangle \\
&= -\Sigma_j \langle \phi^{\#}, R(V_i, V_j)V_j\rangle\langle \omega^i, \phi\rangle \\
&= -\Sigma_j \operatorname{Rm}(\phi^{\#}, V_j, V_j, \phi^{\#}) \\
&= -\operatorname{Ric}(\phi^{\#}, \phi^{\#})
\end{aligned}$$

Our **Weitzenböck Formula for 1-forms** thus reads

$$-\tfrac{1}{2}\Delta|\phi|^2 = -\langle \Delta\phi, \phi\rangle + |\nabla\phi|^2 + \operatorname{Ric}(\phi^{\#}, \phi^{\#}).$$

We finish up this section by exhibiting two more formulas that are generally very useful.

Corollary: (Weitzenböck Formula for Vector Fields)

Let $X \in \mathfrak{X}(M)$. Then if $X^\flat \in \Omega^1(M)$ is closed,

$$-\tfrac{1}{2}\Delta |X|^2 = \langle \nabla(\operatorname{div} X), X\rangle + |\nabla X|^2 + \operatorname{Ric}(X,X).$$

Proof: $X^\flat \in \Omega^1(M)$, and $\flat$ is an isometry, so by the above

$$-\tfrac{1}{2}\Delta|X|^2 = -\tfrac{1}{2}\Delta|X^\flat|^2 = -\langle \Delta X^\flat, X^\flat\rangle + |\nabla X^\flat|^2 + \operatorname{Ric}(X,X).$$

Note that $\flat$ commutes with ∇, so that $|\nabla X^\flat| = |\nabla X|$. Moreover,

$$\Delta X^\flat = d\delta X^\flat + \delta d X^\flat = d\delta X^\flat = -d(\operatorname{div} X)$$

where we use that $\operatorname{div} X = -\delta(X^\flat)$. Thus,

$$-\langle \Delta X^\flat, X^\flat\rangle = \langle d(\operatorname{div} X), X^\flat\rangle = \langle d(\operatorname{div} X)^\sharp, X\rangle$$
$$= \langle \operatorname{grad}(\operatorname{div} X), X\rangle.$$

∎

Corollary (Bochner's Formula)

Let $f \in C^\infty(M)$. Then

$$-\tfrac{1}{2}\Delta|\nabla f|^2 = -\langle \nabla\Delta f, \nabla f\rangle + |\nabla^2 f|^2 + \operatorname{Ric}(\nabla f, \nabla f).$$

Proof: Just notice that ∇f has $d((\nabla f)^\flat) = d(df) = 0$, and apply the previous formula.

∎

Applications of the Weitzenböck Formulas

First, we introduce a few important concepts.

- Killing Fields
- The Maximum Principle
- The Hodge Theorem

And then we prove some results concerning:

- How curvature affects topology;
- The size of isometry groups of Riemannian manifolds;
- Eigenvalue estimates for the Laplacian
- Gårding's Inequality

Killing Fields:

Def: A vector field $X \in \mathfrak{X}(M)$ is a Killing Field iff its local flows act by isometries.

Prop: Let X be a Killing Field. TFAE:

- $\mathcal{L}_X g \equiv 0$ $\qquad \langle \nabla_V X, W \rangle + \langle \nabla_W X, V \rangle = 0$
- $(V \mapsto \nabla_V X)$ is a skew symmetric $(1,1)$ tensor.

Prop: A Killing field X is uniquely determined by its values $X|_p$ and $\nabla X|_p$ at any $p \in M$.

Putting these together, we get:

Theorem: If X is a Killing field, then $\{X = 0\}$ is a disjoint union of totally geodesic submanifolds, each of even co-dimension.

One last result before we move on:

Theorem: The set of Killing fields $\mathfrak{iso}(M,g)$ is a Lie algebra of dim $\le \frac{n(n+1)}{2}$, and $\mathfrak{iso}(M,g)$ is the Lie algebra of $Iso(M,g)$.

Proposition: (Weitzenböck Formula for Killing Fields)

Let $X \in \mathfrak{X}(M)$ be a Killing field. Then

$$-\tfrac{1}{2}\Delta|X|^2 = |\nabla X|^2 - \mathrm{Ric}(X,X).$$

Proof: This can be easily established using the typical technique with local ON normal frames.

It can also be established invariantly, as in Petersen: let $f = \frac{1}{2}|X|^2$.

(i) $\operatorname{grad} f = -\nabla_X X$.

For every V,

$$\langle \operatorname{grad} f, V\rangle = \nabla_V f = \langle \nabla_V X, X\rangle = -\langle \nabla_X X, V\rangle$$

(ii) $\nabla^2 f(V,V) = |\nabla_V X|^2 - Rm(V,X,X,V)$

$$\begin{aligned}
\nabla^2 f(V,V) &= \langle \nabla_V \operatorname{grad} f, V\rangle = -\langle \nabla_V \nabla_X X, V\rangle \\
&= -\{\langle R(V,X)X, V\rangle + \langle \nabla_X\nabla_V X, V\rangle + \langle \nabla_{[V,X]}X, V\rangle\} \\
&= -Rm(V,X,X,V) - \langle \nabla_X \nabla_V X, V\rangle \\
&\quad -\langle \nabla_{\nabla_V X} X, V\rangle + \langle \nabla_{\nabla_X V} X, V\rangle \\
&= |\nabla_V X|^2 - Rm(V,X,X,V) \\
&\quad - \langle \nabla_X \nabla_V X, V\rangle - \langle \nabla_V X, \nabla_X V\rangle \\
&= |\nabla_V X|^2 - Rm(V,X,X,V) \\
&\quad - X\langle \nabla_V X, V\rangle \\
&= |\nabla_V X|^2 - Rm(V,X,X,V)
\end{aligned}$$

∎

One more preliminary concept:

Theorem: (Elliptic (Strong) Maximum Principle)

$\boxed{a^{ij}\xi_i\xi_j \geq 0}$

Let $P = -a^{ij}\partial_i\partial_j - b^i\partial_i$ be a second order elliptic operator on an open $U \subseteq \mathbb{R}^n$, with smooth coefficients.

Suppose $f \in C^\infty(M)$ is a subsolution, i.e., $Pf \leq 0$, on U, then if f attains its maximum on int U, then f is constant. Likewise if $Pf \geq 0$, and f ... minimum ...

Def: A quantity on M is said to be quasi-positive (resp. negative) if it is everywhere non-negative (resp. non-positive) and is strictly positive (resp. negative) at some point.

Theorem (Bochner) (1946): Let M^n be a closed Riem. mfd with non-positive Ricci curvature. Then every Killing field is parallel.

If Ric is quasi-negative, then every Killing field is zero.

Proof: Since $\mathrm{Ric} \leq 0$, the Killing Field Weitzenböck Formula tells us that

$$0 \leq -\tfrac{1}{2}\Delta|X|^2 = \tfrac{1}{2}\operatorname{div}(\operatorname{grad}|X|^2)$$

So that $|X|^2$ is subharmonic on M with respect to the operator

$$\operatorname{div}(\operatorname{grad} f) = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\, g^{ij}\partial_j f).$$

Since M is closed, $|X|^2$ must therefore be constant, so

$$|\nabla X|^2 \equiv \mathrm{Ric}(X,X) \equiv 0$$

Thus X is parallel, and if Ric is negative definite at some point, we also get that $\nabla X|_p = X|_p = 0$, hence $X \equiv 0$. ∎

Alternative Proof: Use Stokes's Theorem (if we have orientability)

$$\begin{cases} \int_M |\nabla X|^2 - \mathrm{Ric}(X,X) = \int_M \tfrac{1}{2}\Delta|X|^2 = 0 \\ |\nabla X|^2 - \mathrm{Ric}(X,X) \geq 0 \end{cases}$$

∎

Corollary: With M^n as above, $\dim(\mathfrak{iso}(M,g)) = \dim(\mathrm{Iso}(M,g)) \leq n$. If Ric is quasi negative, then $\mathrm{Iso}(M,g)$ is finite.

Proof: Recall that $\mathrm{Iso}(M,g)$ is a compact Lie group when M is compact, and that $\mathfrak{iso}(M,g)$ is spanned by Killing fields.

By Bochner's Theorem, every Killing field is parallel, so the linear evaluation map $\mathfrak{iso}(M,g) \to T_pM$ which sends $X \mapsto X|_p$ is injective. Thus, the first statement follows.

For the second part, we see that every Killing field is 0, and so every connected component of $\mathrm{Iso}(M,g)$ is trivial. By compactness, we conclude that $\mathrm{Iso}(M,g)$ is finite if Ric is quasi positive. ∎

Corollary: With (M^n, g) as above, and $p := \dim \mathfrak{iso}(M, g)$, we have the isometric splitting $\tilde{M} = N \times \mathbb{R}^p$.

Proof: We have in hand p linearly independent and parallel vector fields on M, which we can lift to parallel vector fields on $\tilde{M}$.

Fix any $x \in \tilde{M}$. The parallel fields above give a reduction of TM for the action of $\mathrm{Hol}(M, g)$:

$$T\tilde{M} = T^{(0)}\tilde{M} \oplus T^{(1)}\tilde{M} \oplus \cdots \oplus T^{(k)}\tilde{M}$$

where $T^{(0)}\tilde{M}$ is the subbundle spanned by the parallel fields, which is thus acted upon trivially by $\mathrm{Hol}(M, g)$. Note $\dim T^{(0)}M = p$.

Since $\tilde{M}$ is complete and simply connected, the de Rham Decomposition tells us that

$$\tilde{M} =_{\text{isom.}} \mathbb{R}^p \times N.$$

Alternatively: Petersen has a proof using distance functions:

We can make the parallel vector fields on $\tilde{M}$ ON, and then each one arises as the gradient field of a (different) distance function with vanishing Hessians.

This allows us to "split off" Euclidean pieces of the metric

$$\rightsquigarrow \quad g = dr^2 + g_r = dr^2 + g_0$$

We do this for each of the p vector fields, and get the desired splitting. ∎

do we need this?

Theorem: (Bochner, 1948) Let M^n be a closed, oriented Riem. mfd with non-negative Ricci curvature. Then every harmonic 1-form is parallel.

If Ric is quasi-positive, then every harmonic 1-form is zero.

Proof: Let $\phi \in \Omega^1(M)$ be harmonic. Then WF II for Harmonic 1-Forms yields

$$-\tfrac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + \mathrm{Ric}(\phi^\#, \phi^\#) - \cancel{\langle\Delta\phi, \phi\rangle} \geq 0$$

so we see that $|\phi|^2$ is subharmonic, hence constant, and we conclude exactly as before. We could have also used Stokes's Theorem instead of the max. princ. $\blacksquare$

Corollary Let M^n be a closed oriented Riem. mfd with $\mathrm{Ric} \geq 0$.

Then $b_1(M) \leq n$, with equality holding iff (M,g) is a flat torus.

Proof: If $\mathcal{H}^1(M)$ denotes the space of harmonic 1-forms on M, then the Hodge Theorem implies that $b_1(M) = \dim \mathcal{H}^1(M)$.

By Bochner's Theorem, every harmonic 1-form on M is parallel.

Aside: { Conversely, every parallel p-form is closed and co-closed as a result of the expressions
$$d = \omega^i \wedge \nabla_{v_i} \qquad \delta = -\Sigma\, i(v_j)\nabla_{v_j},$$
hence every parallel p-form is harmonic. }

Thus, the linear evaluation map from $\mathcal{H}^1(M) \to T^\#_p M$ which sends $\omega \mapsto \omega_p$ is injective, hence $b_1(M) \leq n$.

Now suppose that equality is achieved, so that there are n independent parallel 1-forms on M. By raising indices we obtain a parallel global frame $\{E_i\}$ for TM. Thus, (M,g) is flat.

Now consider the universal cover $(\tilde{M}, \tilde{g})$ of (M,g), which by flatness is $(\mathbb{R}^n, g_0)$. $\pi_1(M) =: \Gamma$ acts on $\tilde{M} = \mathbb{R}^n$ by isometries.

Lift the frame $\{E_i\}$ to $\{\tilde{E}_i\}$ on $\mathbb{R}^n$, which is again parallel and thus constant. By changing coordinates, we can view $\tilde{E}_i = \partial_i$, the standard coordinate vector fields.

These vector fields are invariant under Γ: $D\gamma_p(\partial_i|_p) = \partial_i|_{\gamma(p)}$. However, this implies that every $\gamma \in \Gamma$ must be a translation, so Γ is f.g., abelian, and torsion free. Thus, $\Gamma = \mathbb{Z}^q$ for some q.

If $q<n$, then $\mathbb{Z}^q$ generates a q-dim. subspace $V \subsetneq \mathbb{R}^n$, and if W is its orthogonal complement then

$$M = \mathbb{R}^n / \mathbb{Z}^q = V \oplus W / \mathbb{Z}^q = V/\mathbb{Z}^q \oplus W$$

contradicting compactness. Thus, $q=n$, so that M is a flat torus.

Remark: If Ric is quasi positive, then $b_1(M) = 0$.

What about the higher Betti numbers?

Def: The curvature operator $R: \Gamma(\Lambda^2 TM) \longrightarrow \Gamma(\Lambda^2 TM)$ is defined via the local formula

$$R(v_i \wedge v_j) = R_{ijkl}\, v_k \wedge v_l$$

where $\{v_i\}$ is a local ON frame as usual.

By the symmetries of R (specifically, $R_{ijkl} = R_{klij}$), we see that R is self adjoint, and therefore has real eigenvalues.

We say that R is non-negative, positive, quasi-positive, etc, if its eigenvalues have that property.

Theorem: Let (M^n, g) be closed and oriented, and let $1 \leq k \leq n-1$.

If $R \geq 0$, then every harmonic k-form is parallel, so

$$b_k(M) \leq \binom{n}{k} = b_k(T^n).$$

If R is quasi-positive, then there are no nontrivial harmonic k-forms, hence

$$b_k(M) = 0.$$

Bohm-Wilking: $R > 0 \rightsquigarrow \begin{cases} S^n \\ \text{Symmetric space} \\ \text{biholo} - \mathbb{C}P^n \end{cases}$

using Ricci flow

Eigenvalue Estimates and Rigidity

Recall that, by some functional analysis, the eigenvalues of $\Delta : C^\infty(M) \to \mathbb{R}$ form an increasing sequence

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \longrightarrow \infty.$$

The corresponding eigenfunctions are, of course, smooth and dense in $L^2(M)$. Let's focus on the eigenvalues for now though:

Theorem: (Lichnerowicz)

Let (M^n, g) be a closed Riem. mfd. with $Ric \geq (n-1)C$ for some $C > 0$. Then

$$\lambda_1 \geq nC.$$

Proof: Let $f \in C^\infty(M)$. Then at a point $x \in M$ at which we center a local ON normal frame $\{V_i\}$,

$$\mathrm{tr}\, \nabla^2 f = (1, \ldots, 1) \cdot (\nabla_{V_1}\nabla_{V_1} f, \ldots, \nabla_{V_n}\nabla_{V_n} f) \leq \sqrt{n}\, |\nabla^2 f|$$

$$\longrightarrow \qquad \frac{1}{n}(\Delta f)^2 \leq |\nabla^2 f|^2.$$

Let f be an eigenfunction of Δ, and apply this estimate to the Bochner Formula:

$$-\frac{1}{2}\Delta|\nabla f|^2 = -\langle \nabla \Delta f, \nabla f\rangle + |\nabla^2 f|^2 + Ric(\nabla f, \nabla f)$$

$$\geq -\lambda |\nabla f|^2 + \frac{1}{n}(\Delta f)^2 + Ric(\nabla f, \nabla f)$$

$$= -\lambda |\nabla f|^2 + \frac{\lambda}{n} f \Delta f + Ric(\nabla f, \nabla f)$$

Recalling Green's Formula

$$\int_M u \Delta v + \int_M \langle \nabla u, \nabla v\rangle = \int_{\partial M} u \langle \nabla v, \nu\rangle \qquad \forall u, v \in C^\infty(M)$$

we can integrate the above estimate over M (which has $\partial M = \emptyset$):

$$0 = -\int_M \Delta |\nabla f|^2 \geq \int_M \left(-\lambda + \frac{\lambda}{n} + (n-1)C\right)|\nabla f|^2$$

So that $\frac{\lambda}{n} + (n-1)C - \lambda \leq 0 \longrightarrow \lambda \geq nC$ as desired. ∎

Theorem: (Obata) Let (M^n, g) be a closed Riem. mfd with $Ric \geq (n-1)C$ for some $C>0$. If $\lambda_1 = nC$, then

$$(M,g) \text{ is isometric to } (S^n(\tfrac{1}{\sqrt{C}}), g_{rd})$$

Proof: WLOG $C=1$, and $\lambda_1 = n$.

In the proof above, we have that

$$Ric(\nabla f, \nabla f) = (n-1)|\nabla f|^2.$$

Recalling that $\Delta(f)^2 = 2f\Delta f - 2|\nabla f|^2$, we obtain, using the Bochner Formula Estimate above, that

$$-\tfrac{1}{2}\Delta(|\nabla f|^2 + f^2) \geq f\Delta f - n|\nabla f|^2 + (n-1)|\nabla f|^2 - f\Delta f + |\nabla f|^2 = 0$$

Since the integral of the LHS over M is 0, we conclude that

$$\Delta(|\nabla f|^2 + f^2) \equiv 0$$

Thus, $|\nabla f|^2 + f^2 \equiv \alpha$ for some constant α.

Now, normalize f so that $\|f\|_\infty = 1$. At a max/min point, $\nabla f = 0$, so that we obtain $\alpha = 1$, and that $\max f = 1 = -\min f$.

Let $p, q \in M$ be s.t. $f(p) = -1$, $f(q) = 1$. Let $\gamma : [0,a] \to M$ be a minimizing geodesic connecting p and q. If $\phi = f \circ \gamma$, then

$$\frac{|\phi'(t)|}{\sqrt{1-\phi(t)^2}} \leq \frac{|\nabla f(\phi(t))|}{\sqrt{1 - f\circ\gamma(t)^2}} \equiv 1$$

so that after integration from 0 to a,

$$\pi \leq a = d(p,q).$$

Thus, $\operatorname{diam}(M,g) \geq \pi$, but since Bonnet-Myer implies $\operatorname{diam}(M,g) \leq \pi$, by rigidity we conclude.

$\blacksquare$

Remark: A more direct proof of Obata's Theorem exists (and is quite beautiful). In fact, it can be rephrased as:

Theorem (Obata, 1962) A complete Riem mfd (M^n, g), $n \geq 2$, admits a nontrivial soln $\phi : M \to \mathbb{R}$ of

$$\operatorname{Hess}\phi = -K\phi g \qquad (K>0)$$

iff it is isometric to $(S^n(\tfrac{1}{\sqrt{K}}), g_{rd})$.

Finally, one last application, this time to PDE's.

Theorem: (Gårding's Inequality)

$\exists c_1, c_2 > 0$ s.t. $\forall \omega \in \Omega^p(M)$

$$(\Delta\omega, \omega) \geq c_1 \|\omega\|_{H^1}^2 - c_2\|\omega\|_{L^2}^2$$

Proof: By WF II,

$$\langle \Delta\omega, \omega\rangle = \tfrac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 - \langle \textstyle\sum_j \omega^i \wedge i(v_j) R(v_i, v_j)\omega, \omega\rangle$$
$$\geq \tfrac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 - a_1|\omega|^2$$

where $a_1 = a_1(M, R) < \infty$ since M is compact.

Integrating over M we obtain

$$(\Delta\omega, \omega) \geq \int_M |\nabla\omega|^2 - a_1 \int_M |\omega|^2 \qquad \text{(weak-coercivity)}$$

$$= c_1\|\omega\|_{H^1}^2 - c_2\|\omega\|_{L^2}^2$$